

AN UNSTABLE NONLINEAR INTEGRODIFFERENTIAL SYSTEM

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ABSTRACT

Consider the nonlinear integrodifferential system

$$u'(t) = -\int_0^{\pi} \alpha(x)T(x,t)dx, \quad T_t = T_{xx} + \eta(x)g(u(t)),$$

with initial-boundary conditions

$$u(0) = u_0, \quad T(x,0) = f(x), \quad T_x(0,t) = T_x(\pi,t).$$

Let α_0 and η_0 be the zeroth Fourier cosine coefficients of α and η . Under certain general assumptions it is known that if $\alpha_0\eta_0 \neq 0$, then $u(t)$ and $T(x,t)$ tend to zero as $t \rightarrow \infty$. We show that when $\alpha_0\eta_0 = 0$ the functions u and T have limits as $t \rightarrow \infty$. These limits are complicated but can be explicitly expressed.

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I. INTRODUCTION.

We shall study the behavior as $t \rightarrow \infty$ of solutions of the nonlinear system

$$(1) \quad u'(t) = -\int_0^{\pi} \alpha(x) T(x, t) dx, T_t = T_{xx} + \eta(x) g(u(t)), \quad (0 < x < \pi, t > 0)$$

where $' = d/dt$. We assume initial-boundary conditions

$$(2a) \quad u(0) = u_0, \quad T(x, 0) = f(x), \quad (0 \leq x \leq \pi)$$

$$(2b) \quad T_x(0, t) = T_x(\pi, t) = 0. \quad (0 < t < \infty)$$

In case $g(u) = \exp(u) - 1$, these equations describe the behavior of a continuous medium nuclear reactor idealized as a slab of length π with insulated faces. The unknown u is the logarithm of reactor power while T represents the difference between the actual and the design-equilibrium temperatures.

We introduce the Fourier coefficients

$$\alpha_0 = (\sqrt{2}/\alpha) \int_0^{\pi} \alpha(s) ds, \alpha_n = (2/\pi) \int_0^{\pi} \alpha(s) \cos ns \, ds$$

for $n = 1, 2, 3, \dots$. Similarly η_n and f_n are the Fourier cosine coefficients of η and f . Define two sequences

$$(3) \quad h_n = \alpha_n \eta_n, \quad k_n = \alpha_n f_n \quad (n = 0, 1, 2, \dots)$$

and two functions

$$(4a) \quad a(t) = (\pi/2) \sum_{n=0}^{\infty} h_n \exp(-n^2 t),$$

$$(4b) \quad b(t) = (\pi/2) \sum_{n=0}^{\infty} k_n \exp(-n^2 t). \quad (0 \leq t < \infty).$$

Then the solution u of (1,2) must satisfy the Volterra integro-differential equation

$$(5) \quad u'(t) = -b(t) - \int_0^t a(t-s)g(u(s))ds, \quad u(0) = u_0.$$

Once the solution of (5) is known, the function $T(x, t)$ is given by

$$(6) \quad T(x, t) = T_0(t)/\sqrt{2} + \sum_{n=1}^{\infty} T_n(t) \cos nx,$$

where for $n = 0, 1, 2, \dots$ we have

$$(7) \quad T_n(t) = \{f_n + \eta_n \int_0^t \exp(n^2 s)g(u(s))ds\} \exp(-n^2 t).$$

Bronikowski [1] studied (1,2) when $g(u) = u$ is linear.

Under certain assumptions he shows that if $h_0 > 0$ the functions

$u(t)$ and $T(x,t)$ decay to zero exponentially as $t \rightarrow \infty$. Using the exponential decay one can prove the local stability of the trivial solution $u \equiv 0$, $T \equiv 0$ of the nonlinear problem (1,2). Levin and Nohel [2] also study the nonlinear problem (1,2). In the stable case $h_0 > 0$ they show that the trivial solution of (1,2) is globally asymptotically stable. They also obtain a result in case $\alpha_0 = \eta_0 = 0$. The purpose of this paper is to obtain complete results in the case where $h_0 = 0$. We prove:

THEOREM 1. Suppose the following assumptions are true:

- (A1) f, f', η, η' and $\alpha \in L^2(0, \pi)$,
- (A2) $h_n \geq 0$ for all n and $h_n > 0$ for at least one $n > 0$,
- (A3) g is locally Lipschitz continuous on $-\infty < u < \infty$,
- (A4) $ug(u) > 0$ if $u \neq 0$,
- (A5) $G(u) = \int_0^u g(s)ds \rightarrow \infty$ as $|u| \rightarrow \infty$,
- (A6) there exists $K > 0$ such that $|g(u)| \leq K(G(u)+1)$ for all u , and
- (A7) $g'(0)$ exists and $g'(0) > 0$.

If $h_0 = k_0 = 0$, then the solutions u and T exist for all $t \geq 0$ and satisfy

$$a) \quad u^{(j)}(t) \rightarrow 0 \quad \text{as } t \rightarrow \infty, \quad (j = 0, 1, 2)$$

- b) $g(u(t)) \in L^1(0, \infty)$, and
- c) the limit $T(x, t)$ exists uniformly in $0 \leq x \leq \pi$ as
 $t \rightarrow \infty$ and this limit equals

$$(f_0 + \eta_0 \int_0^\infty g(u(s)) ds) / \sqrt{2}.$$

THEOREM 2. Suppose $h_0 = 0$, $k_0 \neq 0$ (i.e. $\alpha_0 f_0 \neq 0, \eta_0 = 0$). Let
 (A1-2) be true and suppose g satisfies

(A8) $g \in C^1(-\infty, \infty)$ with $g'(u) > 0$ for all u ,

(A9) for each u_1 there exists $K = K(u_1) > 0$ such that

$$|g(u+u_1) - g(u_1)| \leq K(G(u+u_1) - G(u_1) - u g(u_1) + 1).$$

Define $M = -k_0 / (\sum_{n=1}^\infty h_n n^{-2})$. If there exists u_1 such that $g(u_1) =$
 M , then the solution u and T exist for all $t > 0$ and

a) $u(t) \rightarrow u_1$ as $t \rightarrow \infty$,

b) $u'(t)$ and $u''(t) \rightarrow 0$ as $t \rightarrow \infty$.

c) $\lim_{t \rightarrow \infty} T(x, t) = f_0 / \sqrt{2} + g(u_1) \sum_{n=1}^\infty (\eta_n \cos nx) / n^2$

uniformly for $0 \leq x \leq \pi$.

If $g(u) > M$ ($< M$) for all u , then

d) $u(t) \rightarrow +\infty$ ($-\infty$) as $t \rightarrow \infty$,

e) $\lim_{t \rightarrow \infty} T(x, t) = f_0 / \sqrt{2} + g^* \sum_{n=1}^\infty (\eta_n \cos nx) / n^2$

where $g^* = \lim_{u \rightarrow +\infty} g(u)$ as $u \rightarrow +\infty$ ($-\infty$).

Note that when $g(u) = \exp u - 1$ the conditions (A3), (A4), (A5), (A7) and (A8) are clearly true. It is easily shown that (A6) is true for any $K > e(e-1)^{-1}$. Similarly in (A9) we can take $K > e(e-1)^{-1} \max \{\exp(u_1), 1\}$. For this g we cannot have $M > g(u)$ in Theorem 2. It is possible to have $M < g(u)$, that is $M \leq -1 < \exp(u) - 1$ for all $u \leq 0$.

II. PRELIMINARIES.

We need the following lemma which is a special case of a result of Levin and Nohel [3, Theorem 1].

LEMMA 1: Suppose $g(u)$ satisfies (A3-6) and the functions $a(t)$ and $b(t)$ satisfy

- (i) $a(t) \in C[0, \infty) \cap C^3(0, \infty)$, $(-1)^k a^{(k)}(t) \geq 0$
for $0 \leq t < \infty$, $k = 0, 1, 2, 3$.
- (ii) $a(t) \neq a(0)$,
- (iii) $b(t) \in C[0, \infty) \cap L^1(0, \infty)$, $b'(t) \in C(0, \infty)$ and $|b'(t)|$
is bounded on $0 < t < \infty$.

If $u(t)$ is any solution of equation (5) then $u(t)$ exists for all $t \geq 0$ and $u(t), u'(t) \rightarrow 0$ as $t \rightarrow \infty$.

We shall also need some information concerning the equation obtained from (5) by linearization,

$$(8) \quad v'(t) = -b(t) - \int_0^t a(t-s)g'(0)v(s)ds, v(0) = u_0.$$

We shall study equation (8) using techniques similar to those of [4]. We shall assume throughout the following discussion that (A1), (A2) and (A7) are true and that $h_0 = k_0 = 0$.

Since $a(t)$ and $b(t)$ are bounded, it is easily shown that $v(t)$ exists for all $t \geq 0$, is unique and is of exponential order, c.f. [4, Lemma 4.1] or [5, Theorem 2.1]. Let $V(w)$ denote the Laplace transform

$$V(w) = \int_0^{\infty} \exp(-wt)v(t)dt.$$

We shall show that V satisfies the hypotheses of a Tauberian theorem due to Von Stachó, c.f. [6, p. 277].

An elementary calculation using (4) and (8) shows that

$$(w + (\pi/2)g'(0)I_1(w))V(w) = u_0 - (\pi/2)I_2(w)$$

where

$$I_1(w) = \sum_{n=1}^{\infty} h_n(w+n^2)^{-1}, \quad I_2(w) = \sum_{n=1}^{\infty} k_n(w+n^2)^{-1}.$$

The functions I_j are clearly analytic functions of the complex variable w when $w \neq -1, -4, -9, \dots$. If $\sigma = \operatorname{Re} w \geq 0$, $w \neq 0$, then by (A2)

$$\operatorname{Re} I_1(\sigma + i\tau) = \sum_{n=1}^{\infty} h_n(\sigma + n^2)((\sigma + n^2)^2 + \tau^2)^{-1} > 0.$$

Thus $V(w)$ is analytic when $\operatorname{Re} w \geq 0$.

From the form of I_j it follows by elementary estimates that

$$(9) \quad |I_j^{(n)}(w)| \leq H n! |\tau|^{-n-1}, \quad (w = \sigma + i\tau)$$

for $j = 1, 2$, and $n = 0, 1, 2, \dots$. The constant H is independent of j and n . Also

$$(10) \quad |w + (\pi/2)g'(0)I_1(w)| \geq |\tau| - H|\tau|^{-1}$$

when $w = \sigma + i\tau$, $-\infty < \sigma < \infty$, $|\tau| > 0$.

Using (9) and (10) it follows by induction, c.f. [4, Lemma 5.4], that

$$(11) \quad |u^{(n)}(w)| \leq O(|\tau|^{-n-1}) \quad \text{as } |\tau| \rightarrow \infty,$$

$n = 0, 1, 2, \dots$. The above estimates and integration by parts show that for any σ_0 , $T > 0$

$$(12) \quad \left| \int_y^\infty \exp(i\tau t) V(\sigma_0 + i\tau) d\tau \right| + \left| \int_{-\infty}^{-y} \exp(i\tau t) V(\sigma_0 + i\tau) d\tau \right| \rightarrow 0 \quad (\text{as } y \rightarrow \infty)$$

uniformly for $T \leq t < \infty$. The details of this are the same as those in the proof of Lemma (5.5) of [4].

LEMMA 2. Suppose (A1-2) hold and $v(t)$ solves (8).

- a) If $h_0 = k_0 = 0$, then $t^n v(t) \rightarrow 0$ as $t \rightarrow \infty$ for
 $n = 0, 1, 2, \dots$
- b) If $h_0 = 0$, then the resultant kernel $R(t)$ for equation
 (8) satisfies $t^n R(t) \rightarrow 0$ as $t \rightarrow \infty$ for $n = 0, 1, 2, \dots$

PROOF: Our analysis of $V(w)$ shows that the hypotheses of Von Stachó's Tauberian theorem are satisfied. Thus a) follows immediately. Part b) is a special case of a) since $R(t)$ is the solution of (8) in the special case where $u_0 = 0$ and $b(t) \equiv a(t)g'(0)$.

III. PROOF OF THEOREM 1.

Assumption (A2) and line (4a) imply (i) and (ii) of Lemma 1. Since $k_0 = 0$ line (4b) implies (iii). Thus Lemma 1 applies to the unique solution u of (5). It follows that $u(t)$ exists for all $t \geq 0$ and that both $u(t)$ and $u'(t)$ tend to zero as $t \rightarrow \infty$. Define

$$(13) \quad D(u) = g(u) - g'(0)u, \quad (-\infty < u < \infty)$$

and

$$A(t) = g'(0) \int_0^t a(s)ds, \quad B(t) = \int_0^t b(s)ds, \quad (0 \leq t < \infty)$$

Since $D(u) = o(|u|)$ and $u(t) \rightarrow 0$,

$$|g(u(t))| \leq g'(0)|u(t)| + K_1|u(t)|, \quad (t > 0)$$

for some constant $K_1 > 0$. Thus we will prove that $g(u(t)) \in L^1(0, \infty)$ if we prove that $u(t) \in L^1(0, \infty)$.

The resultant R satisfies

$$-R(t) + A(t) = \int_0^t R(t-s)A(s)ds. \quad (t \geq 0)$$

Since $u(t)$ solves

$$u(t) = u_0 - B(t) - \int_0^t A(t-s)g(u(s))ds/g'(0),$$

it follows that

$$\begin{aligned} u(t) &= u_0(1 - \int_0^t R(s)ds) - (B(t) - \int_0^t R(t-s)B(s)ds) \\ &\quad - \int_0^t R(t-s)D(u(s))ds, \\ &= v(t) - \int_0^t R(t-s)D(u(s))ds. \end{aligned}$$

The function v is the solution of problem (8).

Pick $K > 0$ such that $|D(u(t))| \leq K$ for all $t \geq 0$. By Lemma 2 we may assume for the same K that

$$|R(t)| \leq K(t+1)^{-3}, \quad |v(t)| \leq K(t+1)^{-3}. \quad (t \geq 0)$$

Since $u(t) \rightarrow 0$, there exists $T > 0$ such that for all $t \geq 0$

$$|D(u(t+T))| \leq K^{-1}|u(t+T)|.$$

Thus when $t \geq 0$

$$\begin{aligned} u(t+T) &= v(t+T) - \int_0^T R(t+T-s)D(u(s))ds \\ &\quad - \int_0^t R(t-s)D(u(s+T))ds, \\ |u(t+T)| &\leq K(t+1)^{-3} + K^2(t+1)^{-2} + \\ &\quad + \int_0^t (t+1-s)^{-3}|u(s+T)|ds, \\ &\leq H_1(t) + \int_0^t H_2(t-s)|u(s+T)|ds. \end{aligned}$$

The comparison theorem of Nohel [5, Theorem 2.1] implies that $|u(t+T)| \leq U(t)$ for all $t \geq 0$ where U is the solution of

$$(14) \quad U(t) = H_1(t) + \int_0^t H_2(t-s)U(s)ds.$$

Since H_1 and $H_2 \in L^1(0, \infty)$ and

$$\int_0^\infty H_2(t)dt = 1/2,$$

it follows by the principle of contraction mappings that (14) has a unique solution $U \in L^1(0, \infty)$. Since $U(t)$ majorizes $|u(t+T)|$ we see that $u \in L^1(0, \infty)$. This proves part b) of Theorem 1.

To finish part a) note that

$$u''(t) = -b'(t) - a(0)g(u(t)) - \int_0^t a'(t-s)g(u(s))ds.$$

Since $g(u(t)) \rightarrow 0$ and $a'(t)$ is $L^1(0, \infty)$, the dominated convergence theorem implies

$$\int_0^t a'(t-s)g(u(s))ds \rightarrow 0.$$

Since $b'(t)$ and $g(u(t))$ also tend to 0, we see that $u''(t) \rightarrow 0$ as $t \rightarrow \infty$.

To prove part c) note that $T(x, t)$ is given by formulas (6) and (7). Thus

$$\begin{aligned} |T(x, t) - (f_0 + \eta_0 \int_0^\infty g(u(s))ds)/\sqrt{2}| \leq \\ |(\eta_0/\sqrt{2}) \int_t^\infty g(u(s))ds| + \sum_{n=1}^\infty |f_n| \exp(-t) \\ + \sum_{n=1}^\infty |\eta_n| \int_0^t \exp(s-t) |g(u(s))| ds \rightarrow 0 \end{aligned}$$

as $t \rightarrow \infty$ uniform for x on the interval $0 \leq x \leq \pi$. This proves Theorem 1.

IV. PROOF OF THEOREM 2.

Using the definition of u_1 , α , $a(t)$ and $b(t)$ it follows that

$$u'_0(t) = -b_0(t) - \int_0^t a(t-s)g_0(u_0(s))ds,$$

where

$$u_0(t) = u(t) - u_1, \quad g_0(u) = g(u+u_1) - M,$$

and

$$b_0(t) = (\pi/2) \sum_{n=1}^{\infty} (k_n - M h_n n^{-2}) \exp(-n^2 t).$$

Parts a) and b) of Theorem 2 may now be proved in exactly the same manner as part a) of Theorem 1.

The function $T(x, t)$ is defined by formulas (6) and (7).

Since $u(t) \rightarrow u_1$,

$$\begin{aligned} & \left| \int_0^t \exp(-n^2(t-s)) g(u(s)) ds - g(u_1)/n^2 \right| = \\ & \left| \int_0^t \exp(-n^2(t-s)) (g(u(s)) - g(u_1)) ds + \int_t^{\infty} \exp(-n^2 s) g(u_1) ds \right| \\ & \leq \int_0^t \exp(s-t) |g(u(s)) - g(u_1)| ds + |g(u_1)| \exp(-t) \rightarrow 0. \end{aligned}$$

Thus uniformly for $0 \leq x \leq \pi$ the expression

$$\begin{aligned} & \left| T(x, t) - f_0/\sqrt{2} - \sum_{n=1}^{\infty} (g(u_1) \eta_n \cos nx)/n^2 \right| \leq \\ & \sum_{n=1}^{\infty} (|f_n| \exp(-t) + |\eta_n| \left| \int_0^t \exp(-n^2(t-s)) g(u(s)) ds - g(u_1)/n^2 \right|) \end{aligned}$$

tends to zero as $t \rightarrow \infty$

If $M > g(u)$ for all u (or $M < g(u)$), then it is easily shown that $u(t) \rightarrow +\infty(-\infty)$ and $g(u(t)) \rightarrow g(+\infty)$ (or $g(-\infty)$). The behavior of $T(x,t)$ as $t \rightarrow \infty$ may then be analyzed using the method above. This proves Theorem 2.

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